Generalized Schmidt decomposition based on injective tensor norm

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We present a generalized Schmidt decomposition for a pure system with any number of two-level subsystems. For bipartite systems it gives the Schmidt decomposition, but differs from the well-known three-qubit GSD (Acín *et al*, 2000). The basis is symmetric under the permutation of the parties and is derived from the product state defining the injective tensor norm of the state. The largest coefficient quantifies the quantum correlation of the state. Another coefficient provides a criterion for the presence of an unentangled particle in the state. Remaining coefficients have an information on the applicability to the teleportation and superdense coding when the given quantum state is used as a quantum channel.

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The Schmidt decomposition for bipartite systems [1] is a very important tool in quantum information and quantum computing theories. It shows whether two given states are related by a local unitary transformation [2] or not, which states are applicable for perfect teleportation [3] and superdense coding[4], and whether it is possible to transform a given bipartite pure state to another pure state by local operations and classical communications [5]. Many substantial results have been obtained with the help of the Schmidt normal form and its generalization to the multipartite states is a task of prime importance [6–8].

In this letter we suggest a new approach and impose the following requirements to the multipartite decomposition. First and most important of all, the coefficients of the decomposition should be meaningful and reveal the physical nature of a system. Second, the basis should be clearly defined and, in principle, a method for obtaining it should exist. Third, the decomposition should contain a minimal set of state parameters. The idea of the first postulate reflects the fact that the main advantage of the Schmidt decomposition comes from the physically meaningful set of coefficients.

Thus we are looking for a basis for a product states, which is naturally related to the state, such that the expansion of the state function in this basis gives the physically relevant quantities. We would like to start from the product state that defines the injective tensor norm of a given state. Next we form a uniquely defined set of basis states containing the nearest product state as well as its complimentary orthogonal product states and express the state vector as a linear combination of vectors in the set. The coefficients of the expansion, hereinafter referred to generalized Schmidt decomposition (GSD), exhibit the physically significant properties of pure states. The largest coefficient g is the injective tensor norm of the state. It is a very useful quantity and defines some entanglement measures [9–13]. The other coefficient, *say* h, has an information on the presence or absence of an unentangled particle in a given quantum state. We will show in the following that $h = 0$ is a separability criterion for pure states of a general multi-qubit system [14]. The remaining coefficients reveal the applicability of the quantum state to the teleportation and superdense coding. We will show this by considering general two-qubit and three-qubit, and W-type n-qubit systems whose injective tensor norms were already derived analytically. The decomposition describes two-qubit and GHZ-type three-qubit systems in a similar manner. Furthermore all multi-qubit W-type states have the same description in the GSD expansion.

GSD. Consider n-partite pure systems with the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$. The injective tensor norm $g(\psi)$ of a given n-partite pure state $|\psi\rangle$ is defined as $g(\psi) = \sup |\langle \chi_1 \chi_2 \cdots \chi_n|\psi\rangle|$, where the supremum is over all tuples of vectors $|\chi_k\rangle \in \mathcal{H}_k$ with $\|\chi_k\| = 1$ [15]. The nearest product state $|q\rangle = |q_1\, q_2 \cdots q_n\rangle$ must satisfy stationarity equations [12, 16]

$$\langle q_1 q_2 \cdots \widehat{q_k} \cdots q_n|\psi\rangle = g|q_k\rangle, \quad k = 1, 2, \cdots n \tag{1}$$

where the caret means exclusion. This is a nonlinear eigenvalue problem and, as is often the case, the solution is not single-valued [17, 18]. Hereafter we consider only the solutions for which g is the maximal eigenvalue.

Consider now n-qubit system. For each single-qubit state $|q_k\rangle$ there is, up to arbitrary phase, an unique single-qubit state $|p_k\rangle$ orthogonal to it. Each couple of vectors $|q_k\rangle$ and $|p_k\rangle$ is a basis in the Hilbert space $\mathcal{H}_k$ of kth qubit. From these single-qubit states $|q_k\rangle$ and $|p_k\rangle$ one can form a set of 2^n n-qubit product states which form a basis in the full Hilbert space $\mathcal{H}$. Any vector $|\psi\rangle \in \mathcal{H}$ can be written as a linear combination of vectors in the set. Then from stationarity equations (1) it follows that all the coefficients of the product states $|q_1 \cdots q_{k-1} p_k q_{k+1} \cdots q_n\rangle (k = 1, 2, \cdots n)$ are zero. Thus any pure state can be written in terms of $2^n - n$ product states. Furthermore, the phases of vectors $|p_k\rangle$ are free and we can choose them so that all the coefficients t_k of vectors $|p_1 \cdots p_{k-1} q_k p_{k+1} \cdots p_n\rangle (k = 1, 2, \cdots n)$ be positive. Still we have a freedom to make a phase shift $|p_k\rangle \to e^{2i\pi/(n-1)}|p_k\rangle$ which remains unchanged t_k and g. We use this freedom to vary the phase φ of the component $e^{i\varphi} h|p_1 p_2 \cdots p_n\rangle$ ($h \geq 0$ is understood) within the interval $-\pi/(n-1) \leq \varphi \leq \pi/(n-1)$.

Thus the decomposition has $n+1$ real and $2^n - 2n - 1$ complex parameters. After taking into account the normalization condition, one can show that $2^{n+1} - 3n - 2$ real numbers parameterize the sets of inequivalent pure states [19].

Theorem. The kth qubit is completely unentangled if and only if $h(\psi) = 0$ and $t_i(\psi) = 0$ for $i \neq k$.

Proof. Suppose first qubit is completely unentangled and its state vector is $|q_1\rangle$. We have $|\psi\rangle = |q_1\rangle \otimes |\psi'\rangle$. Let the product state $|q_2 q_3 \cdots q_n\rangle$ be the nearest state of $|\psi'\rangle$. Then GSD of $|\psi'\rangle$ takes the form

$$|\psi\prime\rangle = g'|q_2 q_3 \cdots q_n\rangle + \sum_{i=2}^{n} t'_i |p_2 \cdots p_{i-1} q_i p_{i+1} \cdots p_n\rangle + \cdots + e^{i\varphi'} h' |p_2 p_3 \cdots p_n\rangle. \tag{2}$$

Since the nearest state of the state $|\psi\rangle$ is, up to a phase, the product state $|q_1 q_2 \cdots q_n\rangle$, then $g(\psi) = g',\ h(\psi) = 0,\ t_1(\psi) = h'$ and $t_i = 0, i = 2, 3 ... n$. The inverse is also true. From $h(\psi) = 0$ and $t_i(\psi) = 0$ for $i \neq k$ it follows that all the terms in GSD which do not contain $|q_1\rangle$ vanish and $|\psi\rangle = |q_1\rangle \otimes |\psi'\rangle$. Similarly, theorem is true if any other qubit is unentangled.

Consider now $n = 2, 3, 4$ cases. For simplicity we will use notations $|0_i\rangle$ and $|1_i\rangle$ for vectors $|q_i\rangle$ and $|p_i\rangle$ respectively. Also we will omit sub-indices i whenever it does not create misunderstanding. In the case of two qubit states the expansion reduces to the Schmidt decomposition $|\psi\rangle = g|00\rangle + h|11\rangle$ with $g \geq h \geq 0$. Consider three-qubit case. Decomposition takes the form

$$|\psi\rangle = g|000\rangle + t_1|011\rangle + t_2|101\rangle + t_3|110\rangle + e^{i\varphi} h|111\rangle. \tag{3}$$

The coefficients should satisfy conditions

$$g \geq \max(t_1, t_2, t_3, h),\ t_1 \geq 0,\ t_2 \geq 0,\ t_3 \geq 0,\ h \geq 0,\ -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}. \tag{4}$$

These conditions do not specify GSD uniquely. Eq.(3) is the GSD normal form of the state $|\psi\rangle$ if and only if g is the injective norm. There are highly entangled states which can be written in a form of Eq.(3) in two different bases. One basis, where the largest coefficient is injective tensor norm of the state, gives true GSD while the other, where the largest coefficient is not injective tensor norm, does not. The example with W-type states, which is given below, illustrates this more clearly.

Consider four qubit case. The explicit expression of the expansion is

$$|\psi\rangle = g|0000\rangle + \sum_1^4 t_i |0_i 111\rangle + \sum_1^6 e^{i\varphi_{ij}} t_{ij} |0_i 0_j 11\rangle + e^{i\varphi} h|1111\rangle. \tag{5}$$

The restrictions on coefficients are: $g \geq \max(t_i, t_{ij}, h),\ t_i \geq 0,\ t_{ij} \geq 0,\ h \geq 0,\ \pi/3 \leq \varphi \leq \pi/3$. Again these conditions are insufficient to determine GSD uniquely. Necessary and sufficient condition is that the first coefficient is the injective tensor norm of the state $|\psi\rangle$.

Consider now several interesting examples.

W-type states. Our first example that we shall discuss in detail is a family of four-parametric W-type states [20]

$$|\psi\rangle = a|100\rangle + b|010\rangle + c|001\rangle + d|111\rangle. \tag{6}$$

If one relabels bases vectors $|0_i\rangle \leftrightarrow |1_i\rangle, i = 1, 2, 3$, then one gets exactly the form given by Eq.(3), provided d is the largest coefficient. But it gives GSD normal form only for the slightly entangled states. Otherwise Eq.(6) is not correct GSD.

Injective tensor norm of the state (6) was derived in Ref.[21]. It was shown that it is differently expressed in two different ranges of definition. In highly entangled region parameters (a, b, c, d) form a cyclic quadrilateral and injective tensor norm is expressed in terms of the the circumradius of the quadrangle. In slightly entangled region injective tensor norm is the largest coefficient. Also there are states in between for which both formulae are valid. These states, called second type shared quantum states, separate slightly and highly entangled states and can be ascribed to both types. Another specific states, called first type shared quantum states, are those for which injective tensor norm is a constant and is defined by $g^2 = 1/2$. These states allow perfect quantum teleportation and superdense coding scenario [22].

Highly entangled region is defined by inequalities

$$r_a = a(b^2 + c^2 + d^2 - a^2) + 2bcd > 0, \quad r_b = b(a^2 + c^2 + d^2 - b^2) + 2acd > 0, \tag{7a}$$

$$r_c = c(a^2+b^2+d^2-c^2)+2abd > 0, \quad r_d = d(a^2+b^2+c^2-d^2)+2abc > 0. \tag{7b}$$

The single-qubit states $|q_i\rangle$ in this region are

$$|q_1\rangle = \frac{\sqrt{r_a r_d}|0_1\rangle + \sqrt{r_b r_c}|1_1\rangle}{4S\sqrt{ad+bc}},\ |q_2\rangle = \frac{\sqrt{r_b r_d}|0_2\rangle + \sqrt{r_a r_c}|1_2\rangle}{4S\sqrt{ac+bd}},\ |q_3\rangle = \frac{\sqrt{r_c r_d}|0_3\rangle + \sqrt{r_a r_b}|1_3\rangle}{4S\sqrt{ab+cd}}, \tag{8}$$

where S is the area of the cyclic quadrilateral (a, b, c, d).

The calculation of the coefficients requires advanced mathematical technique. One has to factorize polynomials of degree ten. We would like to suggest a simple way. First one convinces oneself that each factor is a root for the polynomial and next finds the proportionality coefficient in some particular case. The derivation of h is the most complicated out of all coefficients and one can use the hint: if $a = b + c + d$, then $r_b = r_c = r_d = -r_a$. The resulting answer is

$$g = \frac{L}{2S},\ t_1 = \frac{L|r_1|}{4S(ad+bc)},\ t_2 = \frac{L|r_2|}{4S(bd+ac)},\ t_3 = \frac{L|r_3|}{4S(cd+ab)},\ \varphi = \frac{\pi}{2},\ h = \frac{\sqrt{r_a r_b r_c r_d}}{4LS}, \tag{9}$$

where

$$r_1 = b^2+c^2-a^2-d^2,\ r_2 = a^2+c^2-b^2-d^2,\ r_3 = a^2+b^2-c^2-d^2 \tag{10}$$

and $L = \sqrt{(ab+cd)(ac+bd)(ad+bc)}$. In fact, this set gives a fruitful description of the state. The invariant g is expressed in terms of the circumradius of the cyclic quadrangle a, b, c, d and gives geometric and Groverian entanglement measures of the state. First type shared states are defined by $r_1 r_2 r_3 = 0$ and, therefore, one of coefficients t_i must vanish for these states. On the other hand if $r_k = 0$, then $g^2 = 1/2$ and the corresponding state allows teleportation(and dense coding) scenario. For perfect teleportation the receiver should choose kth particle at initial stage in order to perform the task. Thus the coefficients t_i contain an information on the applicability to the teleportation and precisely indicate which particle the receiver should choose. Second type shared states lie on the separating surface $r_a r_b r_c r_d = 0$, i.e $h = 0$. We conclude that $h > 0$ for highly entangled states and $h = 0$ for second type shared states.

To complete the analysis let us consider the remaining slightly entangled case, that is one of quantities r_a, r_b, r_c and r_d should be negative. Consider for example $r_d < 0$ and the remaining possibilities can be treated similarly. In this case the nearest state is $|111\rangle$[21]. In order to obtain GSD one has to simply relabel bases. Then the final GSD coefficients are

$$g = d,\ h = 0,\ t_1 = a,\ t_2 = b,\ t_3 = c. \tag{11}$$

The obvious conclusion is that $h \neq 0$ only for the highly entangled states and identically vanishes for the slightly entangled states. To get confidence let's consider one-parametric n-qubit W-states

$$|\psi\rangle = a\left(|100\cdots0\rangle + |0100\cdots0\rangle + \cdots + |00\cdots010\rangle\right) + b|00\cdots01\rangle. \tag{12}$$

Slightly entangled region is given by $r_n = (n-1)a^2 - b^2 < 0$ [23]. In this region the last product state $|0\cdots01\rangle$ is the nearest separable state and $g = b,\ h = 0$. In highly entangled region $r_n > 0$ and, consequently, $S_n = (n-1)^2a^2 - b^2 > 0$. The constituent states for the closest separable states are respectively

$$|q_1\rangle = \cdots = |q_{n-1}\rangle = \frac{a\sqrt{(n-1)(n-2)}|0\rangle + \sqrt{r_n}|1\rangle}{\sqrt{S_n}}, \quad |q_n\rangle = \frac{\sqrt{(n-1)r_n}|0\rangle + b\sqrt{n-2}|1\rangle}{\sqrt{S_n}}. \tag{13}$$

Straightforward calculation gives

$$g = (1-b^2)^{\frac{n-1}{2}}\left[\frac{n-2}{S_n}\right]^{\frac{n-2}{2}},\ t_n = \sqrt{(n-2)r_n}\left[\frac{r_n}{S_n}\right]^{\frac{n-2}{2}},\ h = b\sqrt{n-1}\left[\frac{r_n}{S_n}\right]^{\frac{n-2}{2}},\ \varphi = \frac{\pi}{n-1}. \tag{14}$$

These expressions have the same meanings as in the three-qubit case. First, $r_n = 0$ forces $g^2 = 1/2$. Second, $g^2 > 1/2$ and $h = 0$ means the state is slightly entangled. Third, $g^2 < 1/2$ and $h = 0$ means $b = 0$ and, therefore, the last qubit is unentangled. Fourth, we conjecture that: all the states with $r_n = 0$ allow the teleportation scenario and the receiver should choose nth qubit. In summary, suggested GSD indicates the applicability to the teleportation and distinguishes the unentangled particles.

GHZ-type states. Consider now the extended GHZ state [24]

$$|\psi\rangle = a|000\rangle + b|001\rangle + c|110\rangle + d|111\rangle \tag{15}$$

which can be rewritten as $k|00q\rangle + k'|11q'\rangle$, where

$$k = \sqrt{a^2+b^2},\ k' = \sqrt{c^2+d^2},\ |q\rangle = \frac{1}{k}\left(a|0\rangle + b|1\rangle\right),\ |q'\rangle = \frac{1}{k'}\left(c|0\rangle + d|1\rangle\right). \tag{16}$$

Injective tensor norm of this state is[18] $g = \max(k, k')$. It suffices to analyze only the case $k \geq k'$ as the opposite case is similar. The nearest state is $|00q\rangle$ and nonzero coefficients of the decomposition are

$$g = k,\ t_3 = \frac{ac+bd}{k},\ h = \frac{|ad-bc|}{k}. \tag{17}$$

This set of GSD coefficients describes the extended GHZ-type states almost in the same way as bipartite systems. Since $g^2 \geq 1/2$, there is no highly entangled region for GHZ-type states. In this sense W-state is more entangled than GHZ-state. When the extended GHZ-state is most entangled, i.e. $g^2 = 1/2$, it is applicable for both teleportation and dense coding [22] and the situation is same in the case of bipartite systems. In contrast to W-type case, there is no region where h is identically zero. Only on condition $ad = bc$ the canonical coordinate h vanishes. Thus if h vanishes, then the state is biseparable and again the same is true for two-qubit systems. The only difference from two-qubit case is that there is an extra term with the coefficient t_3. It shows that the third particle is unentangled when $h = 0$.

We have generalized the Schmidt decomposition for arbitrary composite systems consisting of two-level subsystems. We have calculated the coefficients of the decomposition for generic two-qubit and three-qubit, and one-parametric n-qubit systems explicitly. It is shown that they provide a profound information on the quantum states. The largest coefficient g gives two entanglement measures and together with the last coefficient h clearly distinguishes the states entangled in inequivalent ways. For W-type states there is entire region including a region where the last coefficient h is identically zero. There is no such region for GHZ-type states. Furthermore, isolated zeros of the function h indicate the appearence of the unentangled particles. The coefficients t_i show whether or not a given state is applicable for perfect teleportation(and dense coding) and precisely indicate which particle the receiver should choose at initial stage in order to perform the task. In summary, the explicit construction of GSD for multi-particle systems will provide a deeper insight into the nature of multipartite entanglement.

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